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WHITEHEAD'S ALGORITHM AND THE STRUCTURE OF FREE GROUPS

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In [3], Kapovich and Lustig introduce the notion of a *filling element* of a free group $F(X)$, an element which fixes no point in any very small action of $F(X)$ on a $\mathbb{R}$-tree. Filling elements are the direct free-group-theoretic analogue to the notion of a filling curve on a closed, orientable, hyperbolic surface. Given the long history of close similarities between free groups and surface groups, we naturally expect filling elements to be both (a) common in $F(X)$, and (b) algorithmically identifiable.

The present author has previously verified the fact that filling elements are common via the following theorem.

Theorem 1 ([6, Theorem 3.8]). *Let $F(X)$ be a finitely-generated non-Abelian free group. Then the set of filling elements of $F(X)$ is exponentially generic in $F(X)$.*

We refer the interested reader to [6] for a brief overview of filling elements, genericity, and the analogy to surface groups.

The present results, currently in preparation for publication, concern whether filling elements, and more generally filling subgroups, can be identified via algorithm. We do not yet have a complete algorithm for the full set of filling subgroups, but we can algorithmically identify a large subset of nonfilling subgroups.

Let $F(X)$ be a finitely generated, non-Abelian free group. A subgroup of $F(X)$ is *filling* if as a subgroup it does not fix a point in any very small action of $F(X)$ on an $\mathbb{R}$-tree. Conversely, a subgroup is *nonfilling* if as a subgroup it fixes a point in some $\mathbb{R}$-tree.

A *segment splitting* of $F(X)$ is a decomposition

$$F(X) = H *_{\mathbb{Z}} K$$

of $F(X)$ as an amalgamated product with *vertex subgroups* H and K and an infinite cyclic associated subgroup. A standard Bass-Serre argument shows that such a decomposition corresponds to a simplicial $\mathbb{R}$-tree with very small action by $F(X)$ in which conjugates of H and K fix vertices. Thus subgroups conjugate into H and K are all nonfilling subgroups of $F(X)$. Our main result is that there is an algorithm to identify nonfilling subgroups of this type.

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A detailed version of this paper has been submitted elsewhere.

Theorem 2. *Let $F(X)$ be a finitely generated, non-Abelian free group. There exists an algorithm to determine for any finitely generated subgroup $G \leq F(X)$ whether there exists a segment splitting*

$$F(X) = H *_{\mathbb{Z}} K$$

such that G is conjugate into either H or K. In the case of an affirmative answer, the algorithm also produces the explicit splitting.

Given a segment splitting $F(X) = H *_{\mathbb{Z}} K$, a topological theorem of Bestvina and Feighn shows there exists an automorphism $\phi \in \mathrm{Aut}(F(X))$ and a partition $X = A \sqcup B$ with $|A| \geq 1$ and $|B| \geq 2$ such that $\phi(H) = \langle A, b \rangle$, $\phi(K) = \langle B \rangle$, and $b \in \langle B \rangle$ [1]. We observe that $\langle B \rangle$ is a proper free factor of $F(X)$, and inclusion in a proper free factor is easily algorithmically detectable. Hence the problem reduces to deciding whether a given subgroup G lies in a subgroup $\langle A, b \rangle$ up to automorphism.

Let $G \leq F(X)$ be a finitely generated subgroup. Recall that the *Stallings graph* of G with respect to basis X is a based X-digraph in which the set of labels of reduced based loops coincides exactly with the freely reduced words in X corresponding to elements of G. We denote this Stallings graph by $\Gamma_X(G)$. The Stallings graph provides a geometric model for a subgroup of a free group, and this permits the application of many combinatorial, geometric, and topological techniques in group theory. We refer readers to [4] for an overview of the construction of the Stallings graph and a wide array of examples of the utility of this object in the study of free groups.

An important finding in the study of both the free group $F(X)$ and its group of automorphisms $\mathrm{Aut}(F(X))$ is Gersten's generalization of Whitehead's algorithm. Whitehead's algorithm determines whether two elements $u, v \in F(X)$ differ by an automorphism of $F(X)$; in other words, whether there exists $\phi \in \mathrm{Aut}(F(X))$ such that $\phi(u) = v$. Essential to Whitehead's algorithm is the set of *Whitehead automorphisms*, a particularly simple set of of generators for $\mathrm{Aut}(F(X))$. The algorithm uses these automorphisms to construct a shortest representative of the $\mathrm{Aut}(F(X))$-orbit of an element of $F(X)$, and then to construct *all* shortest representatives of that orbit. It is then clear that two elements of $F(X)$ differ by an automorphism if and only if they share the same set of shortest representatives. We defer to the classical volume by Lyndon and Schupp for further details on Whitehead's algorithm and Whitehead automorphisms [5].

While Whitehead's algorithm extends easily to finite tuples of elements of $F(X)$, the question of how it might be adapted for finitely generated subgroups of $F(X)$ was open for some time. The first solution to this problem was found by Gersten, who observed that the appropriate analogue of "length" of a finitely generated subgroup is the number of edges in its Stallings graph [2]. We say that a finitely generated subgroup $G \leq F(X)$ is *minimal* if $\Gamma_X(G)$ has no more edges than $\Gamma_X(\phi(G))$ for any $\phi \in \mathrm{Aut}(F(X))$. Gersten's adaptation of Whitehead's algorithm gives us a procedure for generating a minimal subgroup in the $\mathrm{Aut}(F(X))$-orbit of G, and from there generating all minimal subgroups in that orbit.

We say that a Stallings graph $\Gamma_X(G)$ of a finitely generated subgroup G of $F(X)$ satisfies *Property (S)* if

- $\Gamma_X(G) = \Gamma_A \vee \Gamma_B$, where Γ_A (resp. Γ_B) is the subgraph induced by the edges labelled by elements of A (resp. B), and
- Γ_B is a cyclic graph.

Property (S) exactly characterizes stallings graphs of subgroups of the form $\langle A, b\rangle$. Recall that an immersion between X-digraphs is a locally injective map preserving vertices, edge labels, and edge orientations. Stallings graphs with immersions into graphs satisfying Property(S) furthermore characterizes exactly the subgroups of $\langle A, b\rangle$.

Proposition 3. *Let $\Gamma_X(G)$ be the Stallings graph of a finitely generated subgroup G of $F(X)$. Let $\Gamma_X(G)$ admit an immersion onto some graph with Property (S). Then some minimal subgroup in the* $\mathrm{Aut}(F(X))$*-orbit of G also admits an immersion onto some graph with Property (S).*

Whitehead's algorithm provides a method for constructing all minimal subgroups in the $\mathrm{Aut}(F(X))$-orbit of a given finitely generated subgroup G. Therefore, to determine whether G is conjugate into some vertex subgroup of a segment splitting of $F(X)$, we need only check each minimal subgroup in the $\mathrm{Aut}(F(X))$-orbit of G for an immersion into a graph with Property (S). This is easily done, and hence yields the algorithm of Theorem 2.

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